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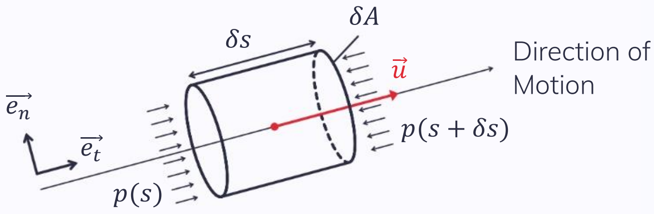
Deriving the Bernoulli Equation

Take an infinitesimally small fluid particle moving along a pathline:



Resolving Tangential Forces

There are three forces acting on the fluid particle:

1. Gravity
2. Pressure
3. Viscous Force – We neglect this one, as we assume inviscid flow

The gravitational force is given as:

$$F_G = mg$$

$$m = \rho v$$

$$v = \delta s \delta A$$

$$F_G = \rho g \delta s \delta A$$

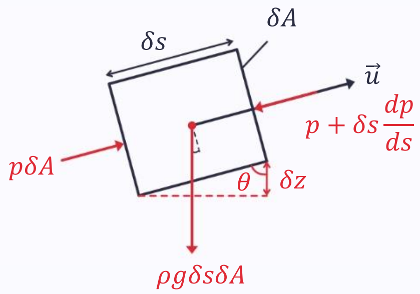
The pressure forces are only taken on the left and right side – it is assumed that the pressure forces around the curved surface will cancel:

$$F = PA$$

$$F_{left} = p(s)\delta A$$

$$F_{right} = p(s+\delta s)\delta A$$

Defining $p(s)$ as p, $p(s+\delta s) = p + \delta s \dfrac{dp}{ds}$



Therefore, the sum of the tangential components:

$$\sum_{\vec{e_t}} F = F_{left} - F_{right} - F_G cos\theta$$

$$\sum_{\vec{e_t}} F = p\delta A - \left(p + \delta s \frac{dp}{ds}\right) \delta A - \rho g \delta s \delta A \, cos\theta$$

To find the angle, $cos\theta$:

$$\delta z = \delta s \, cos\theta$$

$$cos\theta = \frac{dz}{ds}$$

$$\delta z = \delta s \frac{dz}{ds}$$

Therefore:

$$\sum_{\vec{e_t}} F = p\delta A - \left(p + \delta s \frac{dp}{ds}\right) \delta A - \rho g \delta s \delta A \frac{dz}{ds}$$

Finding Acceleration

The particle is travelling with velocity $\vec{u}$. This is given as a function of s only, not t, as the flow is steady:

$$u = u(s) = \frac{ds}{dt}$$

Using the chain rule:

$$a = \frac{du(s)}{dt} = \frac{du}{ds}\frac{ds}{dt}$$

$$a = u\frac{du}{ds}$$

Applying Newton's Second Law

Now that we know the forces, mass and acceleration:

$$F = ma$$

$$p\delta A - \left(p + \delta s \frac{dp}{ds}\right) \delta \mathrm{A} - \rho g \delta s \delta A \frac{dz}{ds} = \rho \delta s \delta A \, u \frac{du}{ds}$$

$$p - \left(p + \delta s \frac{dp}{ds}\right) - \rho g \delta s \frac{dz}{ds} = \rho \delta s \, u \frac{du}{ds}$$

$$-\delta s \frac{dp}{ds} - \rho g \delta s \frac{dz}{ds} = \rho \delta s \, u \frac{du}{ds}$$

$$-\frac{dp}{ds} - \rho g \frac{dz}{ds} = \rho u \frac{du}{ds}$$

If we integrate the term on the right-hand side:

$$\rho u \frac{du}{ds} = \rho \frac{d}{ds}(u \, du) = \rho \frac{d}{ds}\left(\frac{1}{2}u^2\right)$$

$$-\frac{dp}{ds} - \rho g \frac{dz}{ds} = \rho \frac{d}{ds}\left(\frac{1}{2}u^2\right)$$

Now if we assume all the density values are constant, we can divide by density:

$$-\frac{1}{\rho}\frac{dp}{ds} - g\frac{dz}{ds} = \frac{d}{ds}\left(\frac{1}{2}u^2\right)$$

Rearranging:

$$\frac{1}{\rho}\frac{dp}{ds} + \frac{d}{ds}\left(\frac{1}{2}\,u^2\right) + g\frac{dz}{ds} = 0$$

$$\frac{d}{ds}\left(\frac{p}{\rho} + \frac{1}{2}u^2 + gz\right) = 0$$

Integrating with respect to s gives rise to the Bernoulli constant:

$$\frac{p}{\rho} + \frac{1}{2}u^2 + gz = c$$

And there you have it. The Bernoulli Equation.

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