Deriving Reynolds Transport Theorem

Take N as any particular extensive property. Its specific counterpart, therefore, is:

$$\eta = \frac{N}{m}$$

Where m is the mass.

Quantity of Property N
Mass may not be uniform, so can be expressed in terms of a volume integral:

$$m = \int_V \rho \, dV$$

$$N = \int_V \eta\rho \, dV$$

Flow Rate of Property N
From the definition of mass flow rate:

$$\dot{m} = \int_A \rho \, \vec{u}.\overrightarrow{dA}$$

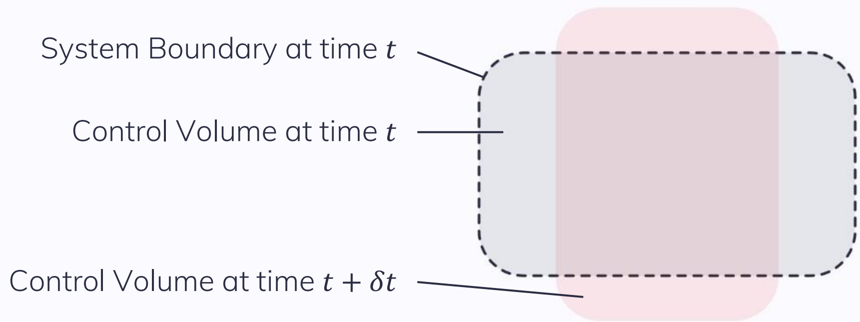
We can find the rate of flow of property N across surface A:

$$\dot{N} = \int_A \eta\rho \, \vec{u}.\overrightarrow{dA}$$

Therefore, the rate of flow of N out of a control volume, CV, is:

$$\dot{N}_{out} - \dot{N}_{in} = \int_A \eta\rho \, \vec{u}.\overrightarrow{dA}$$

Context



The control volume and system boundary are the same at time t. At time $t+\delta t$, however, the control volume has changed. The control surface (the boundary of the control volume) is no longer the same as the system boundary.

> **The idea of the Reynolds Transport Theorem is to relate the rate of change of N in the system to the rate of change of N in the control volume.**

At time $t+\delta t$, the system is made up of two parts:

1. the section that is still in the control volume (the overlapping red and grey area)
2. the two sections on either side that are just grey (in the system, but not in the control volume at $t+\delta t$).

The first of these parts is given as the amount of N in the control volume at $t+\delta t$ minus the amount of N that entered the control volume over δt:

$$N_{CV@t+\delta t} - \dot{N}_{in} \times \delta t$$

The second of these parts is given as the flow of N out of the control volume over time δt:

$$\dot{N}_{out} \times \delta t$$

Therefore, at time $t+\delta t$, the quantity of N in the system is given as:

$$N_{sys@t+\delta t} = N_{CV@t+\delta t} - \dot{N}_{in} \times \delta t + \dot{N}_{out} \times \delta t$$

The change of N in the system then is:

$$N_{sys@t+\delta t} - N_{sys@t} = N_{CV@t+\delta t} - N_{CV@t} - \dot{N}_{in} \times \delta t + \dot{N}_{out} \times \delta t$$

Dividing by δt gives:

$$\frac{1}{\delta t}\left(N_{sys@t+\delta t} - N_{sys@t}\right) = \frac{1}{\delta t}\left(N_{CV@t+\delta t} - N_{CV@t}\right) - \dot{N}_{in} + \dot{N}_{out}$$

Using the definition of differentiation, as $\delta t \to 0$:

$$\frac{dN_{sys}}{dt} = \frac{dN_{CV}}{dt} - \dot{N}_{in} + \dot{N}_{out}$$

Substituting in the quantity of property N from above for the system volume and control volume:

$$\frac{d}{dt}\int_{Vsys(t)} \eta\rho \, dV = \frac{d}{dt}\int_{CV} \eta\rho \, dV - \dot{N}_{in} + \dot{N}_{out}$$

Substituting in the flow rate of property N through the control surface from above:

$$\frac{d}{dt}\int_{Vsys(t)} \eta\rho \, dV = \frac{d}{dt}\int_{CV} \eta\rho \, dV + \int_{CS} \eta\rho \, \vec{u}.\overrightarrow{dA}$$

And there you have it. The Reynolds Transport Theorem.

Return to notes