Fluid Dynamics Key Equations

Steady Streamlines

For a two-dimensional velocity field:

$\vec{u}(x,y) = u(x,y)\hat{\imath} + v(x,y)\hat{\jmath}$

The streamlines are:

$$\int \frac{1}{v(x,y)}\,dy = \int \frac{1}{u(x,y)}\,dx$$

Unsteady Streamlines & Pathlines

For a two-dimensional unsteady field:

$\vec{u}(x,y,t) = u(x,y,t)\hat{\imath} + v(x,y,t)\hat{\jmath}$

The parametrised pathlines are:

$$x = \int u(x,y,t)\,dt \qquad y = \int v(x,y,t)\,dt$$

Forces in Fluids

Shear Stress, τ

$$\tau = \frac{F}{A}$$

$$\tau = \mu\frac{du}{dy}$$

Pressure Force $\quad F_P = PA$

Viscous Force $\quad F_V = \tau A = \mu\frac{du}{dy}A$

Kinematic Viscosity, v $\quad v = \frac{\mu}{\rho}$

Fluid Statics

Hydrostatic Equation

$$\frac{dP}{dz} = -\rho g$$

$$\Delta P = -\rho g \Delta z$$

Resultant Pressure Force

$$F_R = \int_A P\,dA$$

$$F_R = \int_A \rho g y\,dA$$

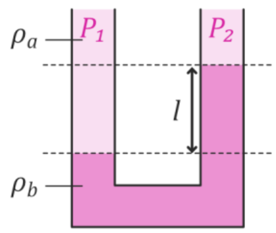
Point of Application

$$y' = \frac{\int_A P y\,dA}{\int_A P\,dA}$$

$$y' = \frac{\int y^2\,dy}{\int y\,dy}$$

- dA must be a function of y

Manometer



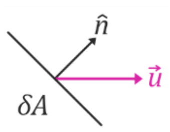
$$P_2 - P_1 = gl(\rho_a - \rho_b)$$

Archimedes' Principle & Buoyancy

"The magnitude of upthrust is equal to the weight of water displaced"

Mass Flow Rate

Vectorial



$$\dot{m} = \int_A \rho\,\vec{u}\,.\vec{dA}$$

Perpendicular $\vec{u}$ $\quad \dot{m} = \rho u A$

Reynold's Transport Theorem

$$\underbrace{\frac{d}{dt}\int_{Vsys(t)} \eta\rho\,dV}_{\text{Rate of change of N in the system}} = \underbrace{\frac{d}{dt}\int_{CV} \eta\rho\,dV}_{\text{Rate of change of N in the CV and system}} + \underbrace{\int_{CS} \eta\rho\,\vec{u}\,.\vec{dA}}_{\text{Net flow rate of N out of the CV}}$$

- N is the property being conserved, $N = \eta m$
- $\rho\eta$ is the property N per unit volume

Conservation of Mass

$$\underbrace{\frac{d}{dt}\int_{CV} \rho\, dV}_{\text{Rate at which the CV gains mass}} = -\underbrace{\int_{CS} \rho\, \vec{u}\,.\,\vec{dA}}_{\text{Net flow rate of mass out of the CV}}$$

Algebraic Formulation

$$\frac{dm}{dt} = \sum \dot{m}_{in} - \sum \dot{m}_{out}$$

Steady Flow

$$\sum \dot{m}_{in} = \sum \dot{m}_{out}$$

$$\int_{CS} \rho\ \vec{u}\,.\,\vec{dA} = 0$$

Steady, uniform flow with constant density

$$\sum uA_{in} = \sum uA_{out}$$

Conservation of Momentum

$$\sum \vec{F} = \frac{d}{dt}\int_{CV} \vec{u}\rho\, dV + \int_{CS} \vec{u}\rho\,\vec{u}\,.\,\vec{dA}$$

- Because $N = M = m\vec{u}$, so $\eta = \vec{u}$

Steady Flow

$$\sum \vec{F} = \int_{CS} \vec{u}\rho\,\vec{u}\,.\,\vec{dA}$$

Steady Flow, resolved into components

$$\sum F_x = \int_{CS} u_x\rho\,\vec{u}\,.\,\vec{dA}$$

$$\sum F_y = \int_{CS} v_y\rho\,\vec{u}\,.\,\vec{dA}$$

Steady, uniform, constant density

$$\sum F_x = \sum_{out} \dot{m}u - \sum_{in} \dot{m}u$$

$$\sum F_y = \sum_{out} \dot{m}v - \sum_{in} \dot{m}v$$

With Conservation of Mass for one inlet and outlet

$$\sum F_x = \dot{m}(u_{out} - u_{in})_x$$

$$\sum F_y = \dot{m}(v_{out} - v_{in})_y$$

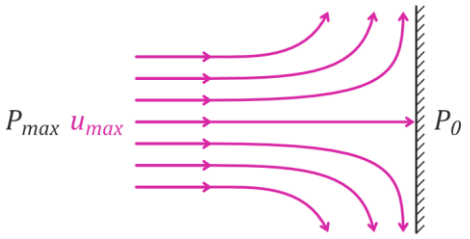
The Bernoulli Equation

$$\frac{P_1}{\rho} + \frac{1}{2}u_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}u_2^2 + gz_2 = c$$

This assumes:

- Steady flow
- Inviscid Flow
- Incompressible (constant ρ) flow
- Two points are on the same streamline/ have the same Bernoulli constant

Stagnation Point Flow



$$\frac{P_{max}}{\rho} + \frac{1}{2}u_{max}^2 = \frac{P_0}{\rho}$$

Conservation of Energy for Steady Flow

$$\dot{Q} - \dot{W} = \int_{CS} \left(\frac{P}{\rho} + \frac{1}{2}u^2 + gz + e\right)\rho\,\vec{u}\,.\,\vec{dA}$$

For a uniform velocity profile:

$$q - w = \Delta\left(\frac{P}{\rho} + \frac{1}{2}u^2 + gz + e\right)$$

1st Thermo Law

$$\frac{dE_{sys}}{dt} = \dot{Q} - \dot{W}$$

Derived from RTP

$$\eta = \frac{1}{2}u^2 + gz + e$$

The Pipe Flow Energy Equation (PFEE)

$$\frac{P_1}{\rho} + \frac{1}{2}u_1^2 + gz_1 = \frac{P_2}{\rho} + \frac{1}{2}u_2^2 + gz_2 + w_L - w_P$$

- w_L is the lost energy
- $w_P = w$ is the pump work

$$\frac{P_1}{\rho g} + \frac{1}{2g}u_1^2 + z_1 = \frac{P_2}{\rho g} + \frac{1}{2g}u_2^2 + z_2 + h_L - h_P$$

- h_L is the lost head
- h_P is the pump head

The pipe flow energy equation only applies for flow that is:

- Steady
- Adiabatic
- Incompressible
- Uniform velocity field
- Between a single inlet and outlet

Laminar Flow between Horizontal Plates

$$u(y) = -\frac{h^2 \Delta P}{2\mu L}\left(1 - \frac{y^2}{h^2}\right)$$

Where u_{max} equals $-\frac{h^2 \Delta P}{2\mu L}$

Laminar Flow in a Circular Pipe

$$u(r) = -\frac{R^2 \Delta P}{4\mu L}\left(1 - \frac{r^2}{R^2}\right)$$

Where u_{max} equals $-\frac{R^2 \Delta P}{4\mu L}$

Turbulent Flow in Circular Pipes

Reynold's Number, Re $\quad Re = \frac{\rho u d}{\mu} = \frac{ud}{v}$

Mean Velocity, u $\quad u = \frac{Q}{A}$

Friction Factor, f $\quad f = \frac{Re}{64}$

Relative roughness, r $\quad r = \frac{\varepsilon}{D}$

Lost Head

Lost head, h_L $\quad h_L = h_f + h_l$

Major Losses, h_f $\quad h_f = f\frac{Lu^2}{2dg}$

Minor Losses, h_l $\quad h_l = k\frac{u^2}{2g}$

Pump Head

Pump head, h_P

$$h_P = \frac{w_P}{g}$$

$$h_P = \frac{\dot{W}_P}{\dot{m}g}$$

$$h_P = \frac{\Delta P_P}{\rho g}$$